

Signed Chord Length Distribution

Part I

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Abstract

In this paper is discussed an application of signed measures (charges) to description of segment and chord length distributions in nonconvex bodies. The signed distribution may naturally appears due to definition via derivatives of nonnegative autocorrelation function simply related with distances distribution between pairs of points in the body. In the work is suggested constructive geometrical interpretation of such derivatives and illustrated appearance of “positive” and “negative” elements similar with usual Hanh–Jordan decomposition in measure theory. The construction is also close related with applications of Dirac method of chords.

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1 INTRODUCTION

Different properties of chord length distribution (CLD) for nonconvex bodies were discussed recently in few publications [1, 2, 3, 4, 5, 6, 7]. Let us recall three different ways to introduce CLD for nonconvex body. Straight line may intersect nonconvex body more than one time (see Fig. 1) and we can either consider each segment of such line as separate chord or calculate sum of all such segments. These two methods are known as multi-chord and one-chord distribution (MCD and OCD) respectively [1, 2, 3].

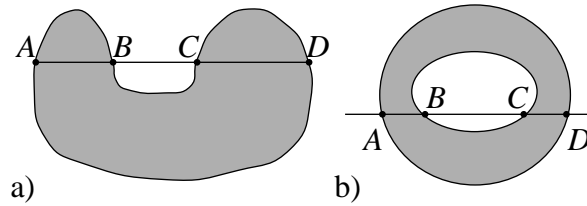


Figure 1: Nonconvex bodies: a) simply connected b) with hole.

For the convex case probability density function for CLD is proportional to second derivative of autocorrelation function [1, 4, 5, 6]. Such a property may be used for a third definition of “the generalized chord distribution” [4, 5, 6], but straightforward calculations demonstrates possibility of negativity of such function for some nonconvex bodies [6]. Let us call this function here *signed chord (length) distribution* to avoid some ambiguity of term “generalized” and to emphasize the basic distinguishing property of this function.

Definition of CLD for convex body has standard *probabilistic interpretation* in theory of *geometric probability* and *random sets* [8, 9, 10]. The MCD and OCD cases for nonconvex body may be described as well [1, 2, 3]. Is it possible to consider similar possibility for *the signed chord distribution*?

In wonderful essay “*Negative probability*” [11] Feynman wrote that, unlike “final probability of a verifiable physical event”, “conditional probabilities and probabilities of imagined intermediate states may be negative” and so: “*If a physical theory for calculating probabilities yields a negative probability for a given situation under certain assumed conditions, we need not conclude the theory is incorrect.*” In this review Feynman provided a few examples with appearance and interpretation of negative probabilities both for quantum and classical physical models.

Mathematical extension of the measure theory for such a purposes may use so-called *signed measures (charges)* [12]. Usually such extension is reduced to standard positive measures due to *Hahn and Jordan decompositions*, corresponding to expression of charge as difference of two positive measures [12].

In many processes with signed distributions the Hahn decomposition, *i.e.*, splitting of space of events on positive and negative parts is quite obvious, *e.g.*, in simplest examples we have two kinds of events: putting and removing objects [11]. A distinction of *the signed chord distribution* is appearance of negativity due to differentiations of positive function without such a natural decomposition on positive and negative elements.

Nonconvex body Fig. 1b may be represented as a convex body and a convex hole and it provides some intuitive justification of possibility to express some distributions using formal difference of convex hull and the hole. Rigor consideration is more difficult, especially for chord distribution expressed via second derivative, *e.g.*, method derived below in Sec. 3.3 reduces examples like Fig. 1 to six “signed” intervals: four “positive”: $[AD]$, $[AB]$, $[CD]$, $[BC]$ and two “negative”: $[AC]$, $[BD]$.

The convex case is revisited in Sec. 2 and Appendix A. The construction of signed chord length distribution for nonconvex body is described in Sec. 3. Some implications to description of arbitrary bodies with nonuniform density are briefly mentioned for completeness in Sec. 4 and Appendix B. Other extensions, like polygonal trajectories are affected very shortly in Sec. 5 and Appendix C.

2 CONVEX BODY

2.1 Basic geometrical models

There are many different functions and relations between them used for description properties of convex bodies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16]. In present paper are considered three different kinds of distributions: distances between points Fig. 2a, lengths of radii (segments) Fig. 2b, and lengths of chords Fig. 2c. It may be useful to describe precisely models of generation of each distribution, to avoid some problems with ambiguity, similar with widely known Bertrand paradox [8, 13].

Definition 1. Distances distribution function is $F_\eta(l) = \int_0^l \eta(x)dx$ with density $\eta(l)$. The distances are defined by pairs of points inside of the body \mathfrak{V} . Both points are from independent uniform distributions, Fig. 2a.

Definition 2. Radii distribution function is $F_\iota(l) = \int_0^l \iota(x)dx$ with density $\iota(l)$. The radii are defined as segments of rays from a point inside of the body \mathfrak{V} to the surface. The points are from uniform distribution and directions of the rays are isotropic, Fig. 2b.

Definition 3. Chord lengths distribution function is $F_\mu(l) = \int_0^l \mu(x)dx$ with density $\mu(l)$. The chords are defined by intersection of the body \mathfrak{V} with isotropic uniform distribution of lines, Fig. 2c.

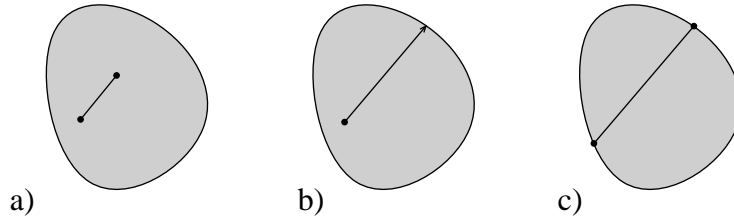


Figure 2: Distributions: a) Distances between points. b) Radii. c) Chords.

It is also convenient to use *autocorrelation function* $\gamma(l)$ related with distances distribution for three-dimensional body as

$$\eta(l) = 4\pi l^2 \gamma(l) / V, \quad (\text{A8 in Appendix A})$$

where V is volume of \mathfrak{V} . The expression for autocorrelation function for body with arbitrary density Eq. (A5) together with derivation of Eq. (A8) for constant density is recollected for completeness below in Appendix A-2.

There is remarkable correspondence between these densities [1, 2, 3, 4, 5, 6, 15, 16]:

$$\frac{1}{\langle l \rangle} \mu(l) = -\iota'(l) = \gamma''(l), \quad (2.1)$$

where average chord length $\langle l \rangle = \int_0^\infty l \mu(l) dl$ may be expressed via volume V and surface area S using a relation for three-dimensional convex body derived in XIX century by Cauchy, Czuber and rediscovered later by Dirac *et al* [8, 10, 14, 15]

$$\langle l \rangle = 4 \frac{V}{S}. \quad (2.2)$$

Distribution of lengths of radii [16] is also known as *interior source randomness* [15]. Relation between $\mu(l)$ and $\iota(l)$ in Eq. (2.1) is often represented in integral form [15, 16]

$$\iota(l) = \langle l \rangle^{-1} \int_l^\infty \mu(x) dx = \langle l \rangle^{-1} (1 - F_\mu(l)).$$

Proportionality between $\mu(l)$ and second derivative of autocorrelation function $\gamma(l)$ in Eq. (2.1) is also well known and widely used [1, 4, 5, 6].

Here is convenient for completeness of presentation and further explanation of nonconvex case to derive all equalities in Eq. (2.1), considering $\mu(l)$, $\iota(l)$, $\gamma(l)$ and $\eta(l)$ as generalized functions.

The usual definition of *generalized function* [12] is a *continuous linear functional* $T(\phi)$ on a space of a *test functions* ϕ . For an integrable function ψ the functional T_ψ is defined for a test function $\phi(x)$ as

$$T_\psi(\phi) = \int_{-\infty}^\infty \psi(x) \phi(x) dx. \quad (2.3)$$

The *generalized derivative* [12] is defined as functional

$$T'(\phi) \equiv -T(\phi'). \quad (2.4)$$

Such a definition ensures derivative of any order for T_ψ with arbitrary integrable function ψ and it justifies use of generalized functions and derivatives in Eq. (2.1).

2.2 Dirac's method of chords

The Dirac's method of chords [14] uses transition from six-dimensional integral over pair of points in some convex body \mathfrak{V} to expressions with chord length distribution, *e.g.*,

$$\mathcal{D}_{\mathfrak{V}}(\varphi) \equiv \frac{1}{V} \int_{\mathfrak{V}} \int_{\mathfrak{V}} \frac{\varphi(R)}{4\pi R^2} d\mathbf{r} d\mathbf{r}' = \frac{S}{4V} \int_0^\infty \mu(x) \left(\int_0^x \int_0^p \varphi(r) dr dp \right) dx, \quad (2.5)$$

where $R = |\mathbf{r}' - \mathbf{r}|$ is distance between points, $d\mathbf{r}$ and $d\mathbf{r}'$ are two three-dimensional volume elements, S is surface area, and V is volume of \mathfrak{V} . Such an equation was derived in Ref. [14] for particular function $\varphi(R) = \exp(-\alpha R)$.

It is shown in Appendix A, that derivation of Eq. (2.5) has direct connection with equalities Eq. (2.1). Here is only outlined basic results and few steps of derivation Eq. (2.5).

First, the integral $\mathcal{D}_{\mathfrak{V}}$ may be expressed via function of distances $\eta(l)$. From Eq. (A2) with Eq. (A4) follows quite understanding relation

$$\frac{1}{V} \mathcal{D}_{\mathfrak{V}}(\varphi) = \frac{1}{V^2} \int_{\mathfrak{V}} \int_{\mathfrak{V}} \frac{\varphi(R)}{4\pi R^2} d\mathbf{r} d\mathbf{r}' = \int_0^\infty \frac{\varphi(x)}{4\pi x^2} \eta(x) dx. \quad (2.6)$$

Due to Eq. (A7) with Eq. (A4) right-hand side of Eq. (2.6) may be rewritten using autocorrelation function $\gamma(l)$ Eq. (A5)

$$\mathcal{D}_{\mathfrak{V}}(\varphi) = \int_0^\infty \gamma(x) \varphi(x) dx \quad (2.7)$$

The six-dimensional integral $\mathcal{D}_{\mathfrak{V}}$ may be reduced to a four-dimensional one by two steps [14] revisited in details in Appendix A. In accordance with *Lemma 1* in Appendix A-1, Eq. (A9) and Eq. (A4) it may be expressed

$$\mathcal{D}_{\mathfrak{V}}(\varphi) = \frac{1}{4\pi V} \int_{\mathfrak{V}} d\mathbf{r} \int d\Omega \int_0^{R_{\max}} \varphi(R) dR, \quad (2.8)$$

where R_{\max} is length of radius for given point and direction, $d\mathbf{r} d\Omega$ is five-dimensional integration on all points and directions. Next, *Lemma 2* in Appendix A-3 let us rewrite Eq. (2.8) using radii density function $\iota(l)$

$$\mathcal{D}_{\mathfrak{V}}(\varphi) = \int_0^\infty \iota(x) \left(\int_0^x \varphi(r) dr \right) dx. \quad (2.9)$$

Finally, due to rather standard arguments [14], revisited in Appendix A-5, it is possible to rewrite Eq. (2.8) using four-dimensional integration on space of lines \mathcal{T}

$$\mathcal{D}_{\mathfrak{V}}(\varphi) = \frac{1}{4\pi V} \int d\mathcal{T} \int_0^{L_{\text{ch}}} dp \int_0^p \varphi(r) dr, \quad (2.10)$$

where $d\mathcal{T}$ is canonical invariant measure on the space of lines and L_{ch} is length of chord for given line intersecting body \mathfrak{V} .

Now it is possible to rewrite Eq. (2.10) using *Lemma 5* from Appendix A-5 and Eq. (A33) to produce initial Eq. (2.5).

These integrals also justify use generalized functions, because may be associated with linear functionals on some test function φ . The relations between integrals may be considered as transformations of these functionals without necessity to indicate any particular φ and it is quite reasonable, because main purpose of this paper is rather discussion about geometrical distributions than about calculation of some integrals.

The Eq. (2.7) may be rewritten using Eq. (2.3) for space of test functions defined on some interval $0 \leq x \leq l_{\max}$

$$\mathcal{D}_{\mathfrak{V}} = T_{\gamma} \equiv \gamma. \quad (2.11a)$$

The Eq. (2.9) may be rewritten due to definition of generalized derivative Eq. (2.4)

$$\mathcal{D}_{\mathfrak{V}}(\phi') = \int_0^\infty \iota(x) \phi(x) dx \implies \mathcal{D}'_{\mathfrak{V}} = -T_{\iota} \equiv -\iota. \quad (2.11b)$$

Finally, from Eq. (2.5) follows

$$\mathcal{D}_{\mathfrak{V}}(\phi'') = \frac{S}{4V} \int_0^\infty \mu(x) \phi(x) dx \implies \mathcal{D}''_{\mathfrak{V}} = \frac{S}{4V} T_{\mu} \equiv \frac{S}{4V} \mu = \frac{1}{\langle l \rangle} \mu. \quad (2.11c)$$

The Eq. (2.11) correspond to Eq. (2.1) for generalized functions and derivatives.

The derivation of integral Eq. (2.7) with autocorrelation function $\gamma(l)$ is quite straightforward and revisited in Appendix A-2 Eq. (A7). The Dirac expression Eq. (2.5) is directly derived from Eq. (2.7) via two integration by parts if Eq. (2.1) is true. So if we could consider Eq. (2.1) as a “definition” of a function $\mu(l)$ via second derivatives of $\gamma(l)$ it ensures Eq. (2.5).

Such a property was used in some works for definition of formal (generalized) chord length distribution via Eq. (2.1) for body with arbitrary shape and density [5, 6].

3 NONCONVEX BODY

3.1 Formal integration by parts

Let us consider application of Eq. (2.5) to some nonconvex body \mathfrak{N} . The distances distribution and autocorrelation function is defined for nonconvex body and so it is possible to write

$$\mathcal{D}_{\mathfrak{N}}(\varphi) \equiv \int_{\mathfrak{N}} \int_{\mathfrak{N}} \frac{\varphi(R)}{4\pi V R^2} d\mathbf{r} d\mathbf{r}' = V \int_0^\infty \frac{\varphi(x)}{4\pi x^2} \eta(x) dx \quad (3.1a)$$

$$= \int_0^\infty \gamma(x) \varphi(x) dx \quad (3.1b)$$

$$= - \int_0^\infty \gamma'(x) \left(\int_0^x \varphi(r) dr \right) dx \quad (3.1c)$$

$$= \int_0^\infty \gamma''(x) \left(\int_0^x \int_0^p \varphi(r) dr dp \right) dx. \quad (3.1d)$$

Here Eq. (3.1a) and Eq. (3.1b) coincide with Eq. (2.6) and Eq. (2.7) in convex case respectively, but Eq. (3.1c) and Eq. (3.1d) are produced by formal integrations by parts and need for geometrical interpretation. Let us denote for nonconvex body the signed radii and chord densities represented via (normalized) first and second derivative as $\iota_{\pm}(l)$ and $\mu_{\pm}(l)$ respectively.

3.2 Radii (signed) density function

An analogue of Eq. (2.8) used for introduction of radii density function has form

$$\mathcal{D}_{\mathfrak{N}}(\varphi) = \frac{1}{4\pi V} \int d\mathbf{r} \int d\Omega \int_{\mathbf{R} \cap \mathfrak{N}} \varphi(R) dR, \quad (3.2)$$

where $\mathbf{R} \cap \mathfrak{N}$ is intersection of a body \mathfrak{N} with a ray from a point inside the body Fig. 3.

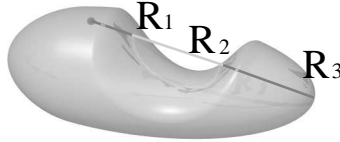


Figure 3: Scheme of intervals for radii in nonconvex body

It is possible to write

$$\int_{\mathbf{R} \cap \mathfrak{N}} \varphi(R) dR \equiv \sum_{j=0}^{n_I} \int_{R_{2j}}^{R_{2j+1}} \varphi(R) dR, \quad R_0 \equiv 0, \quad (3.3)$$

where n_I is amount of intervals $[R_{2j}, R_{2j+1}]$ of ray inside body \mathfrak{N} with R_k for $k > 0$ corresponding to $2n_I - 1$ points of intersection of ray with surface of body Fig. 3. The Eq. (3.3) may be formally rewritten

$$\sum_{j=0}^{n_I} \int_{R_{2j}}^{R_{2j+1}} \varphi(R) dR = \sum_{k=1}^{2n_I-1} (-1)^{k-1} \int_0^{R_k} \varphi(R) dR, \quad (3.4)$$

Let us denote n_{\max} maximal possible amount of intervals for given body and define $\iota_k(l)$, $k = 1, \dots, 2n_{\max} - 1$ as density function for k -th interval length R_k . Then it is possible to derive from Eq. (3.2) for

$$\iota_{\pm}(l) = \sum_{k=1}^{2n_{\max}-1} (-1)^{k-1} \iota_k(l) \quad (3.5)$$

analogue of Eq. (2.9)

$$\mathcal{D}_{\mathfrak{N}}(\varphi) = \sum_{k=1}^{2n_{\max}-1} (-1)^{k-1} \int_0^{\infty} \iota_k(x) \left(\int_0^x \varphi(r) dr \right) dx = \int_0^{\infty} \iota_{\pm}(x) \left(\int_0^x \varphi(r) dr \right) dx \quad (3.6)$$

It is convenient to consider three distributions $\iota_1(l)$, $\iota_+(l)$, and $\iota_-(l)$, where

$$\iota_-(l) = \sum_{k=1}^{n_{\max}-1} \iota_{2k}(l), \quad \iota_+(l) = \sum_{k=1}^{n_{\max}-1} \iota_{2k+1}(l), \quad \iota_{\pm}(l) = \iota_1(l) + \iota_+(l) - \iota_-(l). \quad (3.7)$$

Here “positive” and “negative” distributions, *i.e.*, $\iota_1(l) + \iota_+(l)$ and $\iota_-(l)$ respectively may be nonzero for the same l , *e.g.*, some radii R_2 may be equal to R_3 in other points. So Eq. (3.7) could not be considered as true Jordan decomposition defined as difference of two nonnegative functions with *nonoverlapping* support [12]

$$\iota_{\pm}(l) = \iota^+(l) - \iota^-(l), \quad \forall l : \iota^+(l) \geq 0, \quad \iota^-(l) \geq 0, \quad \iota^+(l) \iota^-(l) = 0. \quad (3.8)$$

In fact, for some nonconvex bodies $\iota_{\pm}(l) \geq 0$ and so $\iota^-(l) \equiv 0$ despite of nonzero $\iota_-(l)$ because of $\iota_1(l) + \iota_+(l) \geq \iota_-(l)$.

Anyway, the scheme discussed above and expressed by Fig. 3, Eq. (3.5), Eq. (3.6), and Eq. (3.7) provides a stochastic interpretation of $\iota_{\pm}(l)$. Similar with *Definition 2* there is an uniform distribution of points inside a body and isotropic rays, considered as some kind of “primary events” for radius R_1 and $\iota_1(l)$. Each ray intersecting body \mathfrak{N} more than one time $n_{\mathfrak{I}} > 1$ also produces two kinds of “secondary” events: $n_{\mathfrak{I}} - 1$ radii R_{2k+1} from a “positive” distribution $\iota_+(l)$ and $n_{\mathfrak{I}} - 1$ radii R_{2k} from a “negative” one $\iota_-(l)$.

Such a stochastic model also produces understanding description of some integrals via averages, mathematical expectations *etc.*, *e.g.*,

$$\int_0^{\infty} \iota_1(l) dl = \int_0^{\infty} \iota_{\pm}(l) dl = 1, \quad \int_0^{\infty} \iota_+(l) dl = \int_0^{\infty} \iota_-(l) dl, \quad (3.9)$$

because each radius corresponds to $n_{\mathfrak{I}}$ “positive” events and $n_{\mathfrak{I}} - 1$ “negative” events, *i.e.*, contribution to “total charge” or “balance” of events $N \equiv N_+ - N_-$ is always one.

It is also useful to introduce distribution of total length of all segments for given radii $\iota_{\mathcal{O}}(l)$ (OSD, one-segment distribution), then

$$\langle l \rangle_{\iota} \equiv \int_0^{\infty} \iota_{\pm}(l) l dl = \int_0^{\infty} \iota_{\mathcal{O}}(l) l dl, \quad (3.10)$$

because contribution for any ray is $\sum_{k=1}^{2n_{\mathfrak{I}}-1} (-1)^{k-1} R_k = R_1 + \sum_{k=1}^{n_{\mathfrak{I}}-1} (R_{2k+1} - R_{2k})$, *i.e.*, total length of all segments.

3.3 Chord length (signed) density function

An analogue of integral Eq. (2.10) for nonconvex body \mathfrak{N} may not use continuous area of integration on $dp dr$, if chord intersects \mathfrak{N} along few intervals. It is similar with integration along set of interval for radii Eq. (3.2) and Eq. (3.3) in Sec. 3.2, but decomposition is more complex because of two integrals.

For a chord with n_I intervals there are $2n_I$ points of intersection ($L_0 \equiv 0, L_1, \dots, L_{2k}, L_{2k+1}, \dots$) for $k = 0, \dots, n_I - 1$. In Eq. (2.10) p is coordinate along the chord and r is distance between points. If to introduce new variables $x = p, x' = p + r$, then for whole chord inside body

$$\blacktriangle^L(\varphi) \equiv \int_0^L \int_0^p \varphi(r) dr dp = - \int_0^L \int_0^{x'} \varphi(x' - x) dx dx' = - \int_0^L \int_x^L \varphi(x' - x) dx' dx, \quad (3.11)$$

but for few segments the integration of $\varphi(r) = \varphi(x' - x)$ on $dx dx'$ should include all points $x, x' \in \mathbf{L} \cap \mathfrak{N}$, $x < x'$, where $\mathbf{L} \cap \mathfrak{N} = \bigcup_{k=0}^{n_I-1} [L_{2k}, L_{2k+1}]$ is union of all intervals of given chord inside the body, Fig. 4a. Here the minus signs in Eq. (3.11) are due to term $-x$ in $\varphi(x' - x)$.

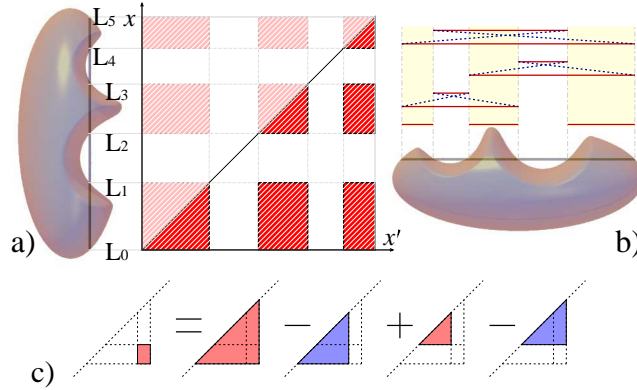


Figure 4: a) Scheme of integration. b) Scheme of intervals c) Decomposition of $\blacksquare_{j,k}$

Let us denote $(\mathbf{L} \cap \mathfrak{N})_{\chi}^2$ area of integration described above and depicted on Fig. 4a as set of triangles and rectangles below of the diagonal $x = x'$

$$(\mathbf{L} \cap \mathfrak{N})_{\chi}^2 = \bigcup_{k=0}^{n_I-1} \blacktriangle_k \cup \bigcup_{k=1, j=0}^{k < n_I, j < k} \blacksquare_{j,k}, \quad (3.12a)$$

$$\blacktriangle_k = \{(x, x') : L_{2k} \leq x \leq x' \leq L_{2k+1}\}, \quad (3.12b)$$

$$\blacksquare_{j,k} = \{(x, x') : L_{2j} \leq x \leq L_{2j+1}, L_{2k} \leq x' \leq L_{2k+1}\}. \quad (3.12c)$$

Now it is possible to write analogue of Eq. (2.10) for nonconvex case

$$\mathcal{D}_{\mathfrak{N}}(\varphi) = \frac{1}{4\pi V} \int \left(- \iint_{(\mathbf{L} \cap \mathfrak{N})_{\chi}^2} \varphi(x' - x) dx dx' \right) dT. \quad (3.13)$$

Due to Eq. (3.12) it may be rewritten

$$\begin{aligned}
-\iint_{(\mathbf{L} \cap \mathfrak{M})_{\times}^2} \varphi(x' - x) dx dx' = \\
= -\left(\sum_{k=0}^{n_1-1} \iint_{\blacktriangle_k} \varphi(x' - x) dx dx' + \sum_{k=1}^{n_1-1} \sum_{j=0}^{k-1} \iint_{\blacksquare_{k,j}} \varphi(x' - x) dx dx' \right), \quad (3.14a)
\end{aligned}$$

where

$$-\iint_{\blacktriangle_k} \varphi(x' - x) dx dx' = \int_0^{L_{2k+1}-L_{2k}} \int_0^p \varphi(r) dr dp = \blacktriangle^{L_{2k+1}-L_{2k}}(\varphi) \quad (3.14b)$$

and integral on $\blacksquare_{j,k}$ may be decomposed on four terms illustrated on Fig. 4c

$$\begin{aligned}
-\iint_{\blacksquare_{j,k}} \varphi(x' - x) dx dx' &= -\int_{L_{2j}}^{L_{2j+1}} \int_{L_{2k}}^{L_{2k+1}} \varphi(x' - x) dx' dx \\
&= \int_{L_{2j}}^{L_{2j+1}} \left(\int_x^{L_{2k}} \varphi(x' - x) dx' - \int_x^{L_{2k+1}} \varphi(x' - x) dx' \right) dx \\
&= \int_0^{L_{2j+1}} \int_x^{L_{2k}} \varphi(x' - x) dx' dx - \int_0^{L_{2j}} \int_x^{L_{2k}} \varphi(x' - x) dx' dx \\
&\quad - \int_0^{L_{2j+1}} \int_x^{L_{2k+1}} \varphi(x' - x) dx' dx + \int_0^{L_{2j}} \int_x^{L_{2k+1}} \varphi(x' - x) dx' dx \\
&= \blacktriangle^{L_{2k}-L_{2j+1}}(\varphi) - \blacktriangle^{L_{2k}-L_{2j}}(\varphi) - \blacktriangle^{L_{2k+1}-L_{2j+1}}(\varphi) + \blacktriangle^{L_{2k+1}-L_{2j}}(\varphi)
\end{aligned} \quad (3.14c)$$

If n_1 is number of segments, there are n_1 (“positive”) terms in Eq. (3.14b) and $4\frac{n_1(n_1-1)}{2}$ terms ($n_1^2 - n_1$ “positive” and $n_1^2 - n_1$ “negative”) in Eq. (3.14c).

It is possible to decompose Eq. (3.13) using Eq. (3.14)

$$\begin{aligned}
\mathcal{D}_{\mathfrak{M}}(\varphi) &= \frac{1}{4\pi V} \int \left(\sum_{k=0}^{n_1-1} \blacktriangle^{L_{2k+1}-L_{2k}}(\varphi) \right. \\
&\quad \left. + \sum_{k=1}^{n_1-1} \sum_{j=0}^{k-1} [\blacktriangle^{L_{2k}-L_{2j+1}}(\varphi) + \blacktriangle^{L_{2k+1}-L_{2j}}(\varphi)] \right. \\
&\quad \left. - \sum_{k=1}^{n_1-1} \sum_{j=0}^{k-1} [\blacktriangle^{L_{2k}-L_{2j}}(\varphi) + \blacktriangle^{L_{2k+1}-L_{2j+1}}(\varphi)] \right) d\mathcal{T}. \quad (3.15)
\end{aligned}$$

Let us recall Eq. (2.10) for convex body expressed using Eq. (3.14b)

$$\mathcal{D}_{\mathfrak{M}}(\varphi) = \frac{1}{4\pi V} \int \blacktriangle^{\mathbf{L}^{\text{ch}}}(\varphi) d\mathcal{T}. \quad (3.16)$$

The Eq. (3.15) with three terms corresponds to sum of $2n_1^2 - n_1$ integrals Eq. (3.16) arranged in three groups. The first one includes all n_1 segments $[L_{2k}, L_{2k+1}]$ of given chord. Second one takes into account $n_1(n_1 - 1)/2$ pairs of intervals $[L_{2j}, L_{2k+1}]$ and $[L_{2j+1}, L_{2k}]$. Third “negative” term is for $n_1(n_1 - 1)/2$ pairs of intervals $[L_{2j}, L_{2k}]$ and $[L_{2j+1}, L_{2k+1}]$.

For a chord with n_1 intervals there are n_1^2 “positive” and $n_1^2 - n_1$ “negative” terms. The scheme is depicted on Fig. 4b for $n_1 = 3$, there “negative” intervals are drawn by dashed lines.

It is possible to decompose $\mu_{\pm}(l)$ on three parts corresponding terms in Eq. (3.16): $\mu_1(l)$, $\mu_+(l)$, $\mu_-(l)$. Here the $\mu_1(l)$ up to normalizing multiplier corresponds to density μ_M of multi-chord distribution (MCD) discussed in Sec. 1.

$$\mu_1(l) = c_M \mu_M(l), \quad c_M = \int_0^\infty \mu_1(l) dl. \quad (3.17)$$

If normalization of $\mu_{\pm}(l)$ is required, it should be expressed

$$\mu_{\pm}(l) = c_M^{-1} [\mu_1(l) + \mu_+(l) - \mu_-(l)] = \mu_M(l) + c_M^{-1} [\mu_+(l) - \mu_-(l)]. \quad (3.18)$$

Here again “positive” $c_M^{-1} [\mu_1(l) + \mu_+(l)]$ and “negative” $c_M^{-1} \mu_-(l)$ terms of Eq. (3.18) can be overlapped and so formally could not be considered as Jordan decomposition [12]

$$\mu_{\pm}(l) = \mu^+(l) - \mu^-(l), \quad \forall l : \mu^+(l) \geq 0, \mu^-(l) \geq 0, \mu^+(l) \mu^-(l) = 0. \quad (3.19)$$

Let us write finally analogue of Eq. (2.5) for nonconvex case

$$\mathcal{D}_{\mathfrak{N}}(\varphi) \equiv \frac{1}{V} \int_{\mathfrak{N}} \int_{\mathfrak{N}} \frac{\varphi(R)}{4\pi R^2} d\mathbf{r} d\mathbf{r}' = \ell_{\mathfrak{N}}^{-1} \int_0^\infty \mu_{\pm}(x) \left(\int_0^x \int_0^p \varphi(r) dr dp \right) dx, \quad (3.20)$$

where $\ell_{\mathfrak{N}}$ is some constant. It was shown earlier for convex body $\ell_{\mathfrak{N}} = \langle l \rangle = 4V/S$.

Definition of *signed chord distribution* as quantity proportional to second derivative of auto-correlation function suggested in Sec. 1 together with comparison of Eq. (3.20) and Eq. (3.1d) produces rather formal equations like

$$\ell_{\mathfrak{N}}^{-1} \mu_{\pm}(l) = \gamma''(l) = -\iota'_{\pm}(l), \quad \ell_{\mathfrak{N}}^{-1} = \int_0^\infty \gamma''(l) dl = \gamma'(0), \quad (3.21a)$$

but Eq. (3.18) also describes “formal” $\mu_{\pm}(l)$ via geometrical distributions of segments of chord in nonconvex body.

It is also possible to write analogue Eq. (A32)

$$\ell_{\mathfrak{N}} = \ell_{\mathfrak{N}} \int_0^\infty \iota_{\pm}(l) dl = -\ell_{\mathfrak{N}} \int_0^\infty \iota'_{\pm}(l) l dl = \int_0^\infty \mu_{\pm}(l) l dl = \langle l \rangle. \quad (3.21b)$$

If to use Eq. (3.20) with $\varphi(l) = 4\pi l^2$ it is possible to derive from Eq. (3.1d) yet another one relation for an average

$$\int_{\mathfrak{N}} \int_{\mathfrak{N}} \frac{d\mathbf{r} d\mathbf{r}'}{V} = \frac{V^2}{V} = \ell_{\mathfrak{N}}^{-1} \int_0^\infty \mu_{\pm}(l) \frac{4\pi l^4}{12} dl \implies \ell_{\mathfrak{N}} = \frac{\pi \langle l^4 \rangle}{3V} \quad (3.21c)$$

It is possible to use stochastic model to clarify some expressions above and derive other useful results. There is uniform isotropic distribution of lines described in *Definition 3* and corresponding to “primary” events. If such line intersects body n_I times, there are $2n_I^2 - n_I$ “secondary” events. There are n_I segments of given chord together with $n_I(n_I - 1)$ “positive” and “negative” intervals described above.

Note on event counting: An essential difference with radii distribution is contribution $n_I \geq 1$ for each chord to “total charge” $N = N_+ - N_-$. So number of lines N_l is not equal with N and $N_l/N \rightarrow c_M$ for $N \rightarrow \infty$. The similar effect is true for work with usual MCD case and so should not be considered as specific difficulty of signed chord distribution.

It is also possible to express c_M using $\mu_O(l)$ density for one-chord case (OCD) also mentioned in Sec. 1, because contribution to average length for μ_1 (without normalization on c_M) is sum of all segments and it is the same for OCD case

$$\int_0^\infty \mu_O(l) l dl = \int_0^\infty \mu_1(l) l dl = c_M \int_0^\infty \mu_M(l) l dl = c_M \int_0^\infty \mu_\pm(l) l dl, \quad c_M = \frac{\langle l \rangle_O}{\langle l \rangle}. \quad (3.22)$$

There is also quite similar expression for $\langle l^2 \rangle$. The contribution to l^2 for one chord due to Eq. (3.15)

$$\begin{aligned} & \sum_{k=0}^{n_I-1} (L_{2k+1} - L_{2k})^2 \\ & + \sum_{k=1}^{n_I-1} \sum_{j=0}^{k-1} ((L_{2k} - L_{2j+1})^2 + (L_{2k+1} - L_{2j})^2 - (L_{2k} - L_{2j})^2 - (L_{2k+1} - L_{2j+1})^2) \\ & = \sum_{k=0}^{n_I-1} (L_{2k+1} - L_{2k})^2 + 2 \sum_{k=1}^{n_I-1} \sum_{j=0}^{k-1} (L_{2k+1} - L_{2k})(L_{2j+1} - L_{2j}) \\ & = \left(\sum_{k=0}^{n_I-1} (L_{2k+1} - L_{2k}) \right)^2 \end{aligned} \quad (3.23)$$

Due to Eq. (3.23) this contribution (to $\mu_1 + \mu_+ - \mu_-$) is always equal to square of sum of all segments, *i.e.*, coincides with OCD case. So, due to Eq. (3.23) and Eq. (3.18)

$$\int_0^\infty \mu_O(l) l^2 dl = \int_0^\infty [\mu_1(l) + \mu_+(l) - \mu_-(l)] l^2 dl = c_M \int_0^\infty \mu_\pm(l) l^2 dl, \quad c_M = \frac{\langle l^2 \rangle_O}{\langle l^2 \rangle}. \quad (3.24)$$

Finally, let us recall few methods for calculation of multiplier $\ell_{\mathfrak{N}}$. It is discussed in Ref. [7] that for convex and nonconvex case $\gamma'(0) = S/4V$ and so due to Eq. (3.21a), it is possible to use $\ell_{\mathfrak{N}} = \gamma'(0)^{-1} = 4V/S$ similar with nonconvex case. There are also interesting results [1, 2] concerning $\langle l \rangle_O$ and $\langle l \rangle_M$ for nonconvex body. It was proven for wide class of nonconvex bodies [2] $\langle l \rangle_M = 4V/S$, and so $\langle l \rangle = \langle l \rangle_M = 4V/S$ and Eq. (3.21b) again ensures $\ell_{\mathfrak{N}} = \langle l \rangle = 4V/S$.

It was also mentioned [2] yet another useful result $\langle l \rangle_O = 4V/S^*$, where S^* is surface area of convex hull and so due to Eq. (3.22) $c_M = S/S^*$. On the other hand, it is not quite clear, if equations with S^* are true for nonconvex body that may not be represented as convex body with few convex holes.

The stochastic model considered here also clarify use Monte-Carlo methods for calculation of signed chord length distribution used in Eq. (3.20). It is necessary to generate uniform isotropic random lines using methods discussed elsewhere [17]. For each chord, intersecting a body n_I times, lengths of all $2n_I^2 - n_I$ segments represented in Eq. (3.15) should be taken into account with proper signs.

It is possible to generate either one signed distribution or two distributions for “positive” and “negative” segments to represent $\mu_\pm(l)$ via difference like Eq. (3.18). It is also necessary to use proper normalization, *i.e.*, use the “total charge” $N = N_+ - N_-$, rather than number of lines N_I (see *Note on event counting* above on page 10).

4 NONUNIFORM CASE

The nonuniform case has more ambiguity in definitions and difficulty with geometrical interpretation. It is only briefly mentioned here. Two analogues of Eq. (2.5) are considered in Appendix B

and have form

$$\iint \rho(\mathbf{r})\rho(\mathbf{r}') \frac{\varphi(\Delta_{\mathbf{r},\mathbf{r}'})}{4\pi|\mathbf{r}-\mathbf{r}'|^2} d\mathbf{r} d\mathbf{r}' = C_\Delta \int_0^\infty \mu_\Delta(x) \left(\int_0^x \int_0^p \varphi(r) dr dp \right) dx, \quad (4.1)$$

where $\Delta_{\mathbf{r},\mathbf{r}'}$ is some function, C_Δ is constant and $\mu_\Delta(x)$ is a density for chosen function $\Delta_{\mathbf{r},\mathbf{r}'}$.

A simple choice $\Delta_{\mathbf{r},\mathbf{r}'} = |\mathbf{r} - \mathbf{r}'|$ is discussed in Appendix B-1 Eq. (B1) and it is proven the correspondence with definition of CLD $\tilde{\mu}(l) \hookrightarrow \mu_\Delta(l)$ via second derivative [5, 6].

In some applications it is more appropriate to consider “optical” length $\Delta_{\mathbf{r},\mathbf{r}'} = \mathcal{O}_{\mathbf{r}}^{\mathbf{r}'}$ defined by Eq. (B4) as an amount of substance along straight line between \mathbf{r} and \mathbf{r}' . For uniform case and convex body it is proportional to distance between points, but for nonuniform case it is another integral Eq. (B5) reproduced in Appendix B-2.

For such an integral there is also analogue of Eq. (2.5). It is expression Eq. (B13) with distribution of “optical” chord lengths $\tilde{\mu}(l) \hookrightarrow \mu_\Delta(l)$ used in *Lemma 6* in Appendix B-2. The complete proof of Eq. (B13) with formal “cancellation” of two terms with ρ may be found in Appendix B-2. This case may be considered as even more direct analogue of convex body with uniform density Eq. (2.5), because $\tilde{\mu}(l)$ is certainly nonnegative due to quite clear geometrical definition.

The $\tilde{\mu}(l)$ is also may be used for nonconvex body \mathfrak{N} with uniform density $\rho = 1$ by formal consideration of union of given body and complement to convex hull with $\rho = 0$. In such a case “optical” length of chord corresponds to sum of all intervals inside \mathfrak{N} , *i.e.*, definition of OCD case [1, 2, 3] mentioned in Sec. 1.

The application of OCD case for nonconvex body with uniform density was suggested already in initial paper about method of chords [14].

5 APPLICATIONS TO ARBITRARY PATHS

A natural extension of problems discussed in present paper — is the consideration of polygonal paths. There are interesting results, concerning trajectories of random walk, *e.g.*, validity of Cauchy formula Eq. (2.2) for average path length inside the body [18, 19, 20, 21, 22]. Appendix C contains a rather illustrative geometrical derivation of this result for uniform isotropic case.

Possibility of application to some tasks concerning arbitrary trajectories is important property of method of chords both for convex Eq. (2.5) and nonconvex case Eq. (3.20). Sometimes consideration of such integrals uses understanding, but simplified models with propagation of particles along straight lines. Such a suggestion is not always justified, because in most cases considered above the segments, radii and chords — are formal subspaces of integration. The only necessary condition is isotropy and uniformity of model, justifying use of function $\varphi(|\mathbf{r}' - \mathbf{r}|)$, instead of some general function $\varphi(\mathbf{r}', \mathbf{r})$.

Formally, the only example of more general function mentioned here is $\varphi(\mathcal{O}_{\mathbf{r}}^{\mathbf{r}'})$ used in Eq. (B13) and discussed in Appendix B-2. For such a model consideration with propagation along line is justified due to definition of $\mathcal{O}_{\mathbf{r}}^{\mathbf{r}'}$ in Sec. 4 and Appendix B-2 Eq. (B4).

Here should be compared two cases. One model with propagation along lines makes possible to use method of chord for convex, nonconvex and even nonuniform case Sec. 4 and Appendix B-2. Other model may be used only for uniform isotropic case, but it is possible to consider also propagation with scattering, that may be described by some function $\varphi(|\mathbf{r}' - \mathbf{r}|)$ due to spherical and translational symmetry Fig. 5.

The basic theme of presented paper is the second model described by signed density function $\mu_\pm(l)$. The first model was discussed for completeness in Sec. 4 and Appendix B-2 and it is shown, that it always described by some nonnegative density function $\tilde{\mu}(l)$ without essential difference in construction for nonconvex and even nonuniform case. The construction of signed chord density function is much complicated and was described in Sec. 3.3.

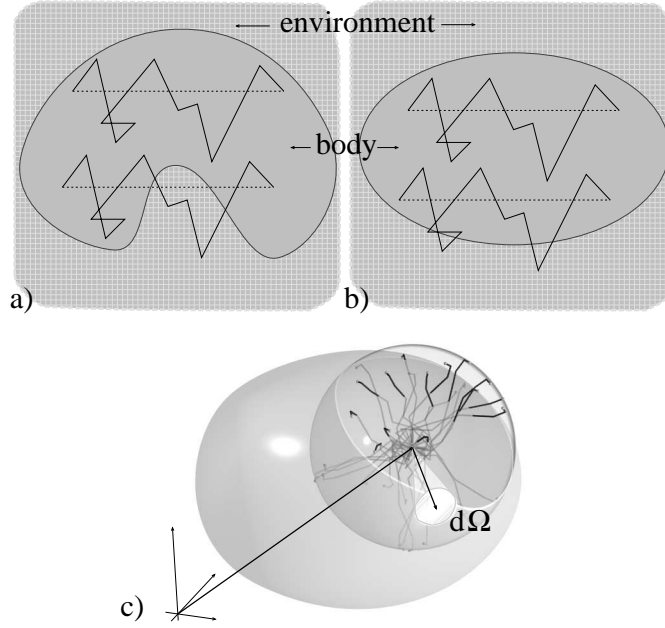


Figure 5: Trajectories and lines: a) Nonconvex body b) Convex body c) Spherical symmetry in distributions of trajectories

Let us consider examples of application of both cases. On Fig. 5 is represented propagation along some trajectories. For nonconvex case Fig. 5a any trajectory may intersect body more than once. For straight line it is always possible to use first model with $\tilde{\mu}(l)$. For nonconvex case and environment with neglectful density $\mathcal{O}_{\mathbf{r}}$, is simply sum of all segments inside body along given line.

A polygonal trajectory may intersect body more than once for both nonconvex and convex cases Fig. 5. If body and environment is the same uniform isotropic media, it is possible to use integrals with $\varphi(|\mathbf{r}' - \mathbf{r}|)$, otherwise there are “boundary effects,” because different trajectories intersect environment by different ways depending on concrete positions \mathbf{r}' and \mathbf{r} even for convex case Fig. 5b.

An illustrative example of an application is a set of random trajectories from uniformly distributed points \mathbf{r} inside the body with isotropic initial directions (see Fig. 5c) and function $\varphi(R) = \varphi(|\mathbf{r}' - \mathbf{r}|)$, treated as probability density of cancellation of trajectory (absorption) on distance R from an origin \mathbf{r} . Then Eq. (2.5) or Eq. (3.20) describe probability for a trajectory of such random walk to be finished inside the body.

APPENDIX A CALCULATION OF DISTRIBUTIONS FOR CONVEX BODY

A-1 Distribution of distances

Let us consider some convex body $\mathfrak{V} \subset \mathbb{R}^3$, function $\Phi(x)$ and integral

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{1}{V^2} \int_{\mathfrak{V}} \int_{\mathfrak{V}} \Phi(|\mathbf{r}' - \mathbf{r}|) d\mathbf{r} d\mathbf{r}', \quad \mathbf{r} = (x, y, z), \quad d\mathbf{r} = dx dy dz, \quad (\text{A1})$$

where V is volume of \mathfrak{V} . It is six-dimensional integral with the function Φ of a distance $|\mathbf{r}' - \mathbf{r}|$ for pair of points $\mathbf{r}, \mathbf{r}' \in \mathfrak{V}$. The multiplier $1/V^2$ is used for normalization $\mathcal{I}_{\mathfrak{V}}(1) = 1$ and Eq. (A1) may be also treated as an average of the function Φ of one variable and expressed via single integration due to lemma below.

Lemma 1. *The linear functional $\mathcal{I}_{\mathfrak{V}}(\Phi)$ defined by Eq. (A1) may be rewritten*

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \int_0^\infty \Phi(x) \eta(x) dx, \quad (\text{A2})$$

where $\eta(x)$ is density of distances distribution in body \mathfrak{V} introduced in Definition 1.

Proof. The linear functional $\mathcal{I}_{\mathfrak{V}}(\Phi)$ also should be considered as a *generalized function* and for regular case it may be represented via integral like Eq. (2.3), i.e., Eq. (A2) with some $\eta(x)$. Let us show, that $\eta(x)$ is density for distribution of distances.

It is enough to consider $\mathcal{I}_{\mathfrak{V}}$ with functions $\Phi(x) = \Theta(l - x) \equiv \Theta_l^-(x)$, there Θ is Heavyside step function, i.e., $\Theta_l^-(x)$ is equal to zero for $x > l$ and unit otherwise. For such a function $\mathcal{I}_{\mathfrak{V}}$ integrates over all pairs of points in $\mathfrak{V} \times \mathfrak{V}$ with distances less than l and due to Eq. (A2) it is possible to write equation for probability $\mathcal{I}_{\mathfrak{V}}(\Theta_l^-) = \mathbb{P}(|\mathbf{r}' - \mathbf{r}| < l) = \int_0^l \eta(x) dx \equiv F_\eta(l)$ and it coincides with definition of density $\eta(x)$ for distances distribution function $F_\eta(x)$.

For extension of the proof for non-regular case, it is possible to use directly the distances distribution function $F_\eta(x)$ and to write instead of Eq. (A2) Lebesgue-Stieltjes integral [12, 23]

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \int_0^\infty \Phi(x) dF_\eta(x). \quad (\text{A3})$$

□

For consideration of both regular and non-regular cases it is also possible to treat η as a generalized function. In such a case representation of $\mathcal{I}_{\mathfrak{V}}(\Phi)$ as an integral Eq. (A2) may look as a not very rigor representation of a tautology like $\mathcal{I}_{\mathfrak{V}} = \eta \equiv T_\eta$, because a generalized function by definition is a linear functional [12].

The functional $\mathcal{I}_{\mathfrak{V}}$ is connected with $\mathcal{D}_{\mathfrak{V}}$ defined in Sec. 2.2, Eq. (2.5) by straightforward relations

$$\mathcal{D}_{\mathfrak{V}}(\varphi) = V \mathcal{I}_{\mathfrak{V}}\left(\frac{\varphi(x)}{4\pi x^2}\right), \quad \mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{1}{V} \mathcal{D}_{\mathfrak{V}}(4\pi x^2 \Phi(x)), \quad (\text{A4})$$

produced by the choice $\Phi(x) = V \frac{\varphi(x)}{4\pi x^2}$.

The meaningfulness *Lemma 1* is not only integral representation Eq. (A2), but association of $\mathcal{I}_{\mathfrak{V}}$ with distribution of distances. It is simple demonstration of some methods to avoid Bertrand-like paradoxes, because $\eta(x)$ is not only possible density for distribution of distances between points in a body.

For calculation of $\mathcal{I}_{\mathfrak{V}}$ Eq. (A1) is used integration on six-dimensional space with natural Euclidean measure on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ and it is in agreement with *independent uniform distributions of both points* used in *Definition 1*.

An example of alternative measure was used in Ref. [16]: it was considered distribution of chords like in *Definition 3* above and segment of line between pair of points on the chords. A distribution function for lengths of the segments, *i.e.*, distances between ending points may be also calculated using Heavyside step function [16], but density is not equivalent with $\eta(x)$.

Yet another example of similar measure may be constructed if we consider first point \mathbf{r} from uniform distribution, but the second one \mathbf{r}' is generated with isotropic distribution of relative directions $\mathbf{R} \equiv \mathbf{r}' - \mathbf{r}$ and uniform distribution of distances $|\mathbf{R}| \equiv |\mathbf{r}' - \mathbf{r}|$ between points. In such a case distributions of points \mathbf{r} and \mathbf{r}' are correlated and relatively to natural Euclidean measure on $(\mathbf{r}, \mathbf{r}') \in \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6 \supset \mathfrak{V} \times \mathfrak{V}$ here is necessary to introduce multiplier $|\mathbf{R}|^{-2} = |\mathbf{r}' - \mathbf{r}|^{-2}$.

Up to constant normalizing multiplier the alternative density of distances introduced above may be expressed as $\eta(x)/x^2$, *i.e.*, proportional to autocorrelation function [1, 4, 5, 6] discussed further.

A-2 Autocorrelation function

For body with density $\rho(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3$ is defined the autocorrelation function $\gamma(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3$ or $\gamma(l)$, $l \in \mathbb{R}$

$$\gamma(\mathbf{r}) = \int_{\mathbb{R}^3} \rho(\mathbf{r}') \rho(\mathbf{r} + \mathbf{r}') d\mathbf{r}', \quad \gamma(l) = \frac{1}{4\pi l^2} \int_{\mathbb{S}_l} \gamma(\mathbf{r}) d\Omega, \quad d\Omega = \sin \theta d\theta d\phi, \quad (\text{A5})$$

i.e., $\gamma(l)$ is an average of $\gamma(\mathbf{r})$ on sphere with radius l , $\{\mathbb{S}_l : |\mathbf{r}| = l\}$.

In simplest case of constant density $\rho(\mathbf{r}) = 1$ for $\mathbf{r} \in \mathfrak{V}$ and zero otherwise. For such a case $\gamma(0) = V$, and it is convenient to rescale $\varrho(\mathbf{r}) \equiv \rho(\mathbf{r})/\sqrt{V}$ for normalization $\gamma(0) = 1$. It is possible to rewrite Eq. (A1)

$$\begin{aligned} \mathcal{I}_{\mathfrak{V}}(\Phi) &= \frac{1}{V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varrho(\mathbf{r}) \varrho(\mathbf{r}') \Phi(|\mathbf{r}' - \mathbf{r}|) d\mathbf{r} d\mathbf{r}' \\ &= \frac{1}{V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varrho(\mathbf{r}) \varrho(\mathbf{r} + \mathbf{R}) \Phi(|\mathbf{R}|) d\mathbf{r} d\mathbf{R} \quad (\mathbf{R} = \mathbf{r}' - \mathbf{r}) \\ &= \frac{1}{V} \int_{\mathbb{R}^3} \gamma(\mathbf{R}) \Phi(|\mathbf{R}|) d\mathbf{R} \end{aligned} \quad (\text{A6})$$

$$= \frac{4\pi}{V} \int_0^\infty l^2 \gamma(l) \Phi(l) dl, \quad (\text{A7})$$

where Eq. (A7) is produced from Eq. (A6) by integration over spheres \mathbb{S}_l .

Comparison of Eq. (A7) and Eq. (A2) with arbitrary function $\Phi(l)$ produces relation between $\gamma(l)$ and $\eta(l)$

$$\eta(l) = \frac{4\pi}{V} l^2 \gamma(l) \quad (\text{A8})$$

already mentioned earlier in Sec. 2.1.

A-3 Distribution of radii

Let us introduce new variable $\mathbf{R} = \mathbf{r}' - \mathbf{r}$ in Eq. (A1) and rewrite integral on $d\mathbf{R}$ using spherical coordinates

$$\begin{aligned}\mathcal{I}_{\mathfrak{V}}(\Phi) &= \frac{1}{V^2} \int_{\mathfrak{V}} d\mathbf{r} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{R(\mathbf{r},\theta,\phi)} R^2 \Phi(R) dR \\ &= \frac{1}{V^2} \int_{\mathfrak{V}} d\mathbf{r} \int_{\mathbb{S}} d\Omega \int_0^{R(\mathbf{r},\Omega)} R^2 \Phi(R) dR,\end{aligned}\quad (\text{A9})$$

where $R^2 \sin\theta$ is Jacobian in spherical coordinates (R, θ, ϕ) of vector $\mathbf{r}' - \mathbf{r}$, \mathbb{S} is unit sphere, and $R(\mathbf{r}, \theta, \phi) = R(\mathbf{r}, \Omega)$ is length of radius (segment) mentioned in *Definition 2*, i.e., distance from point \mathbf{r} to surface of body in direction represented by spherical angles θ, ϕ or unit vector $\Omega \in \mathbb{S}$, Fig. 6.

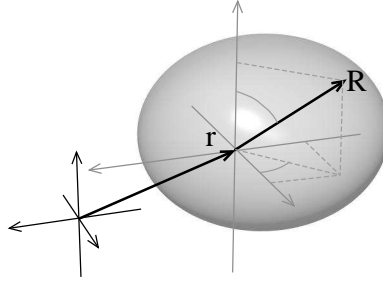


Figure 6: Scheme of integration along radii with length $R(\mathbf{r}, \theta, \phi) = |\mathbf{R}|$

Let us introduce linear functional

$$\mathcal{I}_{\mathfrak{V}}^{(l)}(\Psi) = \frac{1}{4\pi V} \int_{\mathfrak{V}} \int_{\mathbb{S}} \Psi[R(\mathbf{r}, \Omega)] d\mathbf{r} d\Omega, \quad (\text{A10})$$

where $\mathbf{r} \in \mathfrak{V} \subset \mathbb{R}^3$, $\Omega \in \mathbb{S}$, and $R(\mathbf{r}, \Omega)$ is notation for length of radius introduced in Eq. (A9). Here $\mathcal{I}_{\mathfrak{V}}^{(l)}(1) = 1$ and $4\pi V$ is normalization, i.e., 5D volume of set $\mathfrak{V} \times \mathbb{S}$ with respect to canonical measure on $\mathbb{R}^3 \times \mathbb{S}$. It is analogue of normalization of Eq. (A1) with V^2 , i.e., 6D volume of set $\mathfrak{V} \times \mathfrak{V}$ with respect to canonical measure on $\mathbb{R}^3 \times \mathbb{R}^3$.

Lemma 2. The linear functional $\mathcal{I}_{\mathfrak{V}}^{(l)}(\Phi)$ defined by Eq. (A10) may be rewritten

$$\mathcal{I}_{\mathfrak{V}}^{(l)}(\Psi) = \int_0^\infty \Psi(x) \iota(x) dx \quad (\text{A11})$$

where $\iota(x)$ is density of radii distribution in body \mathfrak{V} introduced in *Definition 2*.

Proof. The proof is similar with *Lemma 1*. The Eq. (A11) is again almost tautology for generalized functions $\mathcal{I}_{\mathfrak{V}}^{(l)} = \iota \equiv T_l$. Let us anew consider $\mathcal{I}_{\mathfrak{V}}^{(l)}$ with Heavyside step function $\Theta_l^-(x)$, to integrate over points in $\mathfrak{V} \times \mathbb{S}$ associated with radii less than given l .

The measures of integration in $\mathcal{I}_{\mathfrak{V}}^{(l)}$ correspond to uniform distribution of points \mathbf{r} and isotropic distribution of directions Ω on unit spheres (due to transition to spherical coordinates in second integral). It is in agreement with *Definition 2* and so $\mathcal{I}_{\mathfrak{V}}^{(l)}(\Theta_l^-) = P(R(\mathbf{r}, \Omega) < l) = \int_0^l \iota(x) dx \equiv F_l(l)$ and meets definition of density $\iota(x)$ for radii length distribution function $F_l(x)$. \square

The functional $\mathcal{I}_{\mathfrak{V}}^{(l)}$ Eq. (A10) let us simplify Eq. (A9)

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \mathcal{I}_{\mathfrak{V}}^{(l)} \left(\frac{4\pi}{V} \int_0^x R^2 \Phi(R) dR \right) \quad (\text{A12})$$

and substitution of Eq. (A12) to Eq. (A11) produces yet another expression for $\mathcal{I}_{\mathfrak{V}}(\Phi)$

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \int_0^\infty \left(\frac{4\pi}{V} \int_0^x R^2 \Phi(R) dR \right) \iota(x) dx. \quad (\text{A13})$$

A-4 Autocorrelation function and distribution of radii

Due to definition Eq. (A8) it is possible to rewrite Eq. (A2)

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \int_0^\infty \Phi(x) \frac{4\pi}{V} x^2 \gamma(x) dx. \quad (\text{A14})$$

Yet another expression

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = - \int_0^\infty \left(\frac{4\pi}{V} \int_0^x R^2 \Phi(R) dR \right) \gamma'(x) dx, \quad (\text{A15})$$

may be produced from Eq. (A14) using integration by parts. The equality of Eq. (A15) and Eq. (A13) corresponds to second equation in Eq. (2.1)

$$-\iota(x) = \gamma'(x). \quad (\text{A16})$$

The integral representations above let us also consider Eq. (A16) as derivative of generalized function.

A-5 Chord length distribution

Let us change order of integration in Eq. (A9)

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{1}{V^2} \int_{\mathbb{S}} d\Omega \int_{\mathfrak{V}} d\mathbf{r} \int_0^{R(\mathbf{r}, \Omega)} R^2 \Phi(R) dR \quad (\text{A17})$$

and for any unit vector $\Omega \in \mathbb{S}$ to decompose spatial integral $\int d\mathbf{r} = \iint dP dn$ on two integrals: along the axis $\mathbf{n} = \mathbf{n}_\Omega$ parallel to Ω and on the plane $P = P_\Omega$ perpendicular to Ω Fig. 7.

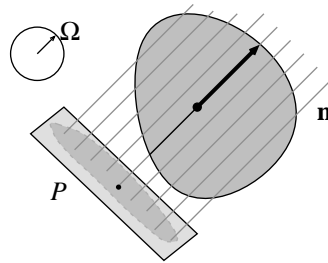


Figure 7: Lines and chords

After such decomposition Eq. (A17) may be rewritten as

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{1}{V^2} \int_{\mathbb{S}} d\Omega \int_{P_{\Omega}(\mathfrak{V})} dP \int_0^{L(p_{\Omega}, \Omega)} dn \int_0^n R^2 \Phi(R) dR, \quad (\text{A18})$$

where $P_{\Omega}(\mathfrak{V})$ is projection of \mathfrak{V} on plane P_{Ω} and $L(p_{\Omega}, \Omega)$ is length of chord of line defined by direction $\Omega \in \mathbb{S}$ and point $p_{\Omega} \in P_{\Omega}$, see Fig. 7.

It is possible to define four-dimensional space \mathcal{T} of (directed) lines [8, 9, 10, 24, 25, 26], there each line is defined by point on sphere $\Omega \in \mathbb{S}$ and point of intersection p_{Ω} with orthogonal plane $P_{\Omega} \perp \Omega$. Here each line is represented twice with two opposite directions $\pm\Omega$. It is possible to introduce space of undirected lines as a quotient space $\tilde{\mathcal{T}} = \mathcal{T}/\{+1, -1\}$, but in most expressions below for convenience of calculations is used \mathcal{T} . The first two integrals in Eq. (A18) correspond to integration on this space

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{1}{V^2} \int_{\mathcal{T}(\mathfrak{V})} d\mathcal{T} \int_0^{L(l_{\mathcal{T}})} dL \int_0^L R^2 \Phi(R) dR, \quad (\text{A19})$$

where $\mathcal{T}(\mathfrak{V})$ is set of lines intersecting \mathfrak{V} , $L(l_{\mathcal{T}})$ is chord length for a line $l_{\mathcal{T}} \in \mathcal{T}$, and $d\mathcal{T}$ is canonical (uniform and isotropic) measure on \mathcal{T} unique defined up to constant multiplier by invariance with respect to translations and rotations [8, 9, 10, 26].

Let us introduce linear functional

$$\mathcal{I}_{\mathfrak{V}}^{(\text{II})}(\Upsilon) = \frac{1}{V[\mathcal{T}(\mathfrak{V})]} \int_{\mathcal{T}(\mathfrak{V})} \Upsilon[L(l_{\mathcal{T}})] d\mathcal{T}, \quad V[\mathcal{T}(\mathfrak{V})] = \int_{\mathcal{T}(\mathfrak{V})} d\mathcal{T}, \quad (\text{A20})$$

where $V[\mathcal{T}(\mathfrak{V})]$ is normalization, *i.e.*, 4D volume of $\mathcal{T}(\mathfrak{V})$ with respect to $d\mathcal{T}$. It may be written also due to definition of $\int_{\mathcal{T}(\mathfrak{V})} d\mathcal{T}$ above

$$V[\mathcal{T}(\mathfrak{V})] = \int_{\mathbb{S}} d\Omega \int_{P_{\Omega}(\mathfrak{V})} dP = 4\pi \langle S_{P_{\Omega}(\mathfrak{V})} \rangle \quad (\text{A21})$$

where $\langle S_{P_{\Omega}(\mathfrak{V})} \rangle$ is average surface of projection of body \mathfrak{V} . For convex bodies due to a Cauchy formula [8, 9]

$$\langle S_{P_{\Omega}(\mathfrak{V})} \rangle = \frac{1}{4} S, \quad (\text{A22})$$

where S is surface area of \mathfrak{V} and so

$$V[\mathcal{T}(\mathfrak{V})] = \pi S. \quad (\text{A23})$$

Lemma 3. *The linear functional $\mathcal{I}_{\mathfrak{V}}^{(\text{II})}(\Upsilon)$ defined by Eq. (A20) may be rewritten*

$$\mathcal{I}_{\mathfrak{V}}^{(\text{II})}(\Upsilon) = \int_0^{\infty} \Upsilon(x) \mu(x) dx \quad (\text{A24})$$

where $\mu(x)$ is density of chord length distribution in body \mathfrak{V} introduced in Definition 3.

Proof. Similar with Lemma 1 and Lemma 2 the Eq. (A24) may be represented in rather trivial form $\mathcal{I}_{\mathfrak{V}}^{(\text{II})} = \mu \equiv T_{\mu}$ for generalized functions. Here again the Heavyside step function $\Theta_L^-(x)$ selects subset in $\mathcal{T}(\mathfrak{V})$ associated with chords shorter than given L . The measure of integration $d\mathcal{T}$ in $\mathcal{I}_{\mathfrak{V}}^{(\text{II})}$ corresponds to uniform and isotropic distribution of lines $l_{\mathcal{T}}$ in agreement with Definition 3 and so $\mathcal{I}_{\mathfrak{V}}^{(\text{II})}(\Theta_L^-) = \text{P}(L(l_{\mathcal{T}}) < L) = \int_0^L \mu(x) dx \equiv F_{\mu}(L)$ and meets definition of density $\mu(x)$ for chord length distribution function $F_{\mu}(x)$. \square

The space of lines has nontrivial structure and an essential moment in proof of *Lemma 3* is correspondence of measure of integration $d\mathcal{T}$ in Eq. (A20) and measure describing distribution of lines in *Definition 3*. This isotropic uniform distribution sometimes associated with concept of μ -randomness [1, 6, 15, 19].

An alternative distribution of lines is often defined by uniform distribution of points inside the body \mathfrak{V} and isotropic distribution of directions. It is sometimes called ν -randomness [1, 19] or interior radiator randomness [15]. It is different from μ -randomness, because the same line may be represented by any point on this line and direction along this line. So measure of ν -chord for given line in comparison with uniform case has extra multiplier proportional to length of the chord $\nu(l) \propto l\mu(l)$. The precise expression for convex body is [1, 15, 19]

$$\nu(l) = \frac{l}{\langle l \rangle} \mu(l) = \left(\frac{4V}{S} \right)^{-1} l \mu(l). \quad (\text{A25})$$

The concept of ν -randomness for chords is close with *Definition 2* of radii distribution and it is useful for some applications [15, 19].

Yet another distribution is produced by definition of line by pair of points with independent uniform distributions inside \mathfrak{V} . It is sometimes called λ -randomness [1, 6, 19]. The measure for λ -chord for three-dimensional case $\lambda(l) \propto l^4 \mu(l)$. The normalizing multiplier for l^4 may be directly calculated [1, 3, 10, 14, 19]

$$\langle l^4 \rangle = \frac{12V^2}{\pi S} \quad (\text{A26})$$

and precise expression for convex body is [1, 19]

$$\lambda(l) = \frac{l^4}{\langle l^4 \rangle} \mu(l) = \left(\frac{12V^2}{\pi S} \right)^{-1} l^4 \mu(l). \quad (\text{A27})$$

The concept of λ -randomness for chords has certain relation with *Definition 1* of distances distribution.

Such different kinds of randomness illustrates necessity of rather pedantic work with distribution of lines due to analogues of Bertrand paradox [8, 13] already mentioned earlier.

Lemma 4. *The functional Eq. (A20) may be formally expressed via Eq. (A10)*

$$\mathcal{I}_{\mathfrak{V}}^{(\text{II})} \left(\int_0^x \Psi(l) dl \right) = \frac{4V}{S} \mathcal{I}_{\mathfrak{V}}^{(\text{I})}(\Psi). \quad (\text{A28})$$

Proof. The derivation of Eq. (A28) is similar with transition from Eq. (A9) to Eq. (A19) via Eq. (A17) and Eq. (A18)

$$\begin{aligned} \frac{4V}{S} \mathcal{I}_{\mathfrak{V}}^{(\text{I})}(\Psi) &= \frac{4V}{S} \frac{1}{4\pi V} \int_{\mathfrak{V}} \int_{\mathbb{S}} \Psi[\mathbf{R}(\mathbf{r}, \boldsymbol{\Omega})] d\mathbf{r} d\boldsymbol{\Omega} = \frac{1}{\pi S} \int_{\mathbb{S}} d\boldsymbol{\Omega} \int_{\mathfrak{V}} d\mathbf{r} \Psi[\mathbf{R}(\mathbf{r}, \boldsymbol{\Omega})] \\ &= \frac{1}{\pi S} \int_{\mathbb{S}} d\boldsymbol{\Omega} \int_{P_{\Omega}(\mathfrak{V})} dP \int_0^{L(P_{\Omega}, \boldsymbol{\Omega})} dn \int_0^n \Psi(l) dl = \mathcal{I}_{\mathfrak{V}}^{(\text{II})} \left(\int_0^x \Psi(l) dl \right). \end{aligned}$$

□

An application Eq. (A24) and Eq. (A11) to Eq. (A28) produces

$$\int_0^\infty \left(\int_0^x \Psi(l) dl \right) \mu(x) dx = \frac{4V}{S} \int_0^\infty \Psi(x) \iota(x) dx. \quad (\text{A29})$$

An integration by parts of Eq. (A29) produces

$$\int_0^\infty \left(\int_0^x \Psi(l) dl \right) \mu(x) dx = -\frac{4V}{S} \int_0^\infty \left(\int_0^x \Psi(l) dl \right) \iota'(x) dx. \quad (\text{A30})$$

The first equality in Eq. (2.1) follows from Eq. (A30)

$$\iota'(x) = - \left(\frac{4V}{S} \right)^{-1} \mu(x) = -\frac{1}{\langle l \rangle} \mu(x). \quad (\text{A31})$$

It may be considered as generalized derivative due to Eq. (A29).

In Eq. (A31) was used Cauchy relation Eq. (2.2) $4V/S = \langle l \rangle$. It is possible also to find normalizing multiplier n_μ for $n_\mu \mu(x) = -\iota'(x)$ simply using integration by parts

$$n_\mu^{-1} = n_\mu^{-1} \int_0^\infty \iota(l) dl = -n_\mu^{-1} \int_0^\infty \iota'(l) l dl = \int_0^\infty l \mu(l) dl = \langle l \rangle. \quad (\text{A32})$$

This derivation uses only the fact of proportionality of $\mu(x)$ to derivative of another density function and in further applications for nonconvex case normalization with $\langle l \rangle$ may be preferable due to nontrivial proof of expression with volume and surface area.

Lemma 5. *The functional $\mathcal{I}_{\mathfrak{V}}$ Eq. (A1) may be expressed via $\mathcal{I}_{\mathfrak{V}}^{(\text{II})}$ Eq. (A20)*

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{S}{4V} \mathcal{I}_{\mathfrak{V}}^{(\text{II})} \left(\int_0^x \left(\int_0^l \frac{4\pi}{V} r^2 \Phi(r) dr \right) dl \right). \quad (\text{A33})$$

Proof. The expression Eq. (A33) is straightforward combination of Eq. (A12) and Eq. (A28). \square

Application of Eq. (A24) to Eq. (A33) produces

$$\mathcal{I}_{\mathfrak{V}}(\Phi) = \frac{S}{4V} \int_0^\infty \left(\int_0^x \int_0^l \frac{4\pi}{V} r^2 \Phi(r) dr dl \right) \mu(x) dx. \quad (\text{A34})$$

The Eq. (A1) and Eq. (A34) produce for $\Phi(r) = V^2 \frac{\varphi(r)}{4\pi r^2}$

$$\int_{\mathfrak{V}} \int_{\mathfrak{V}} \frac{\varphi(|\mathbf{r}' - \mathbf{r}|)}{4\pi |\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' = \frac{S}{4} \int_0^\infty \mu(x) \left(\int_0^x \int_0^p \varphi(r) dr dp \right) dx, \quad (\text{A35})$$

in agreement with Eq. (2.5) in Sec. 2.2. It is also convenient sometimes to use expression

$$\int_0^x \int_0^p \varphi(r) dr dp = \int_0^x (x - r) \varphi(r) dr. \quad (\text{A36})$$

APPENDIX B SOME EQUATIONS FOR NONUNIFORM CASE

Let us consider a body with nonuniform density $\rho(\mathbf{r})$. Any such body \mathfrak{N} may be treated as a convex one without lost of generality by consideration of the convex hull \mathfrak{V} and assignment of zero density to the complement $\mathfrak{V} \setminus \mathfrak{N}$.

Here is considered two cases corresponding to choice of $\Delta_{\mathbf{r}, \mathbf{r}'}$ in Eq. (4.1).

B-1 Distance between points $\Delta_{\mathbf{r},\mathbf{r}'} = |\mathbf{r}' - \mathbf{r}|$

A direct analogue of Eq. (A7) is

$$\begin{aligned} \iint \rho(\mathbf{r})\rho(\mathbf{r}') \frac{\varphi(|\mathbf{r}' - \mathbf{r}|)}{4\pi|\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' &= \int_0^\infty \gamma(x)\varphi(x)dx \\ &= \int_0^\infty \gamma''(x) \left(\int_0^x \int_0^p \varphi(r)drdp \right) dx, \end{aligned} \quad (\text{B1})$$

there second equality produced by two integrations by parts. So up to normalization with $\int_0^\infty \gamma''(x)dx = \gamma'(0)$ it is possible to use a formal (“generalized” [5, 6]) chord length distribution $\dot{\mu}(l) = \gamma''(l)/\gamma'(0)$ for calculation of integrals like Eq. (B1).

Finally

$$\iint \rho(\mathbf{r})\rho(\mathbf{r}') \frac{\varphi(|\mathbf{r}' - \mathbf{r}|)}{4\pi|\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' = \dot{C}_\mu \int_0^\infty \dot{\mu}(x) \left(\int_0^x \int_0^p \varphi(r)drdp \right) dx, \quad \dot{C}_\mu = \gamma'(0). \quad (\text{B2})$$

There is yet another way to express \dot{C}_μ . If to consider $\varphi(l) = 4\pi l^2$, then from Eq. (B2) follows

$$M^2 = \iint \rho(\mathbf{r})\rho(\mathbf{r}') d\mathbf{r} d\mathbf{r}' = \dot{C}_\mu \int_0^\infty \dot{\mu}(x) \left(\int_0^x \int_0^p 4\pi r^2 drdp \right) dx = \frac{\pi}{3} \dot{C}_\mu \int_0^\infty x^4 \dot{\mu}(x) dx.$$

where $M \equiv \int \rho(\mathbf{r})d\mathbf{r}$ is mass of the body. Let us also denote $\langle l^4 \rangle \equiv \int_0^\infty x^4 \dot{\mu}(x) dx$, then

$$\dot{C}_\mu = \frac{3M^2}{\pi \langle l^4 \rangle}. \quad (\text{B3})$$

For case of constant unit density $M = V$ and due to Eq. (A26) $\langle l^4 \rangle = 12V^2/(\pi S)$. So $C_\mu = S/4$ in agreement with Eq. (A35) and Cauchy formula Eq. (A22).

B-2 “Optical” length $\Delta_{\mathbf{r},\mathbf{r}'} = \mathcal{O}_{\mathbf{r}}^{\mathbf{r}'}$

In some applications instead of distance between points $R = |\mathbf{r}' - \mathbf{r}|$ in $\varphi(R)$ it is necessary to use an “optical width”, *i.e.*, integral on density along line between points

$$\mathcal{O}_{\mathbf{r}}^{\mathbf{r}'} = \int_{\mathbf{r}}^{\mathbf{r}'} \rho d\ell = |\mathbf{r}' - \mathbf{r}| \int_0^1 \rho(\mathbf{r} + (\mathbf{r}' - \mathbf{r})x) dx \quad (\text{B4})$$

and to consider functional

$$\mathcal{J}(\varphi) = \iint \rho(\mathbf{r})\rho(\mathbf{r}') \frac{\varphi(\mathcal{O}_{\mathbf{r}}^{\mathbf{r}'})}{4\pi|\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' \quad (\text{B5})$$

Let us introduce new variable and rewrite integral on $d\mathbf{R}$ using spherical coordinates like in Eq. (A9)

$$\mathcal{J}(\varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d\mathbf{r} \int_{\mathbb{S}} d\Omega \rho(\mathbf{r}) \int_0^{R(\mathbf{r},\Omega)} \rho(\mathbf{r} + x\Omega) \varphi(\mathcal{O}_{\mathbf{r}}^{\mathbf{r}+x\Omega}) dx. \quad (\text{B6})$$

where $R(\mathbf{r}, \Omega)$ is lenght of (maximal) radius, $\mathcal{O}_{\mathbf{r}}^{\mathbf{r}+x\Omega} = \int_0^x \rho(\mathbf{r} + l\Omega) dl \equiv s(x)$, $s'(x) = \rho(\mathbf{r} + x\Omega)$. Last integral has form $\int_0^l s'(x)\rho(s(x))dx$ and may be rewritten as $\int_0^{s(l)} \rho(s)ds$.

$$\mathcal{J}(\varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d\mathbf{r} \int_{\mathbb{S}} d\Omega \rho(\mathbf{r}) \int_0^{w(\mathbf{r},\Omega)} \varphi(w) dw, \quad (\text{B7})$$

where $w(\mathbf{r}, \mathbf{\Omega}) = \mathcal{O}_{\mathbf{r}}^{\mathbf{r}+\mathbf{R}(\mathbf{r}, \mathbf{\Omega})\mathbf{\Omega}} = \int_0^{\mathbf{R}(\mathbf{r}, \mathbf{\Omega})} \rho(\mathbf{r} + l\mathbf{\Omega})dl$ is “optical” radius length.

It is possible to use analogue of integration in Eq. (A18) with axis $\mathbf{n} \parallel \mathbf{\Omega}$ and plane $P \perp \mathbf{\Omega}$

$$\mathcal{J}(\varphi) = \frac{1}{4\pi} \int_{\mathbb{S}} d\Omega \int_P dP \int dn \rho(\mathbf{r}) \int_0^{w(\mathbf{r}, \mathbf{\Omega})} \varphi(w)dw, \quad \mathbf{r} = \mathbf{r}_P + n\mathbf{\Omega}, \quad (\text{B8})$$

where point \mathbf{r}_P is intersection (projection) of line representing path of integration with plane P (see Fig. 7) and range of n corresponds to variation of \mathbf{r} along whole chord in convex span of a body.

If to consider last integral in Eq. (B8) as some function $F(w(n))$, two last integrals have form $\int w'(n)F(w(n))dn = \int F(w)dw$ and so we have analogue of Eq. (A19)

$$\mathcal{J}(\varphi) = \frac{1}{4\pi} \int_{\mathcal{T}} d\mathcal{T} \int_0^{W(l_{\mathcal{T}})} dw \int_0^x \varphi(x)dx, \quad (\text{B9})$$

where $W(l_{\mathcal{T}}) = \mathcal{O}_{\mathbf{r}_{\min}}^{\mathbf{r}_{\max}}(l_{\mathcal{T}})$ is “optical” chord length.

Lemma 6. *Let us now introduce “optical” chord length distribution $\tilde{\mu}(x)$, i.e., to any chord of line intersecting (convex hull of) a body in two points \mathbf{r}_1 and \mathbf{r}_2 instead of $|\mathbf{r}_2 - \mathbf{r}_1|$ is assigned “optical” length $\mathcal{O}_{\mathbf{r}_1}^{\mathbf{r}_2}$. The integral Eq. (B5) may be expressed*

$$\mathcal{J}(\varphi) = \tilde{C}_{\mu} \int_0^{\infty} \tilde{\mu}(x) \left(\int_0^x \int_0^p \varphi(r)dr dp \right) dx. \quad (\text{B10})$$

Proof. The Eq. (B10) follows from Eq. (B9) and it is complete analogue of derivation Eq. (A34) from Eq. (A19) in Appendix A. In both cases is used the same integral of some function $F(l_{\mathcal{T}})$ over space of lines \mathcal{T} with the same measure $d\mathcal{T}$ and particular method of calculation of F does not matter. The $\mu(l)$ in Eq. (A34) is (joint) density for $F(l_{\mathcal{T}}) = \mathbf{L}(l_{\mathcal{T}})$ and the $\tilde{\mu}(l)$ in Eq. (B10) is (joint) density for $F(l_{\mathcal{T}}) = W(l_{\mathcal{T}})$.

There is unessential difficulty with constant multiplier \tilde{C}_{μ} , due to lack of simple analogue of Cauchy formula Eq. (A22) for average surface used in normalization of the integral on \mathcal{T} in Eq. (A23). An alternative way of calculation of \tilde{C}_{μ} is represented below. \square

Let us introduce quantity

$$G \equiv \iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{4\pi|\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' = \int_0^{\infty} \gamma(x)dx \quad (\text{B11})$$

then for Eq. (B10) with $\varphi(l) = 1$

$$G = \tilde{C}_{\mu} \int_0^{\infty} \tilde{\mu}(x) \left(\int_0^x \int_0^p dr dp \right) dx = \frac{1}{2} \tilde{C}_{\mu} \int_0^{\infty} x^2 \tilde{\mu}(x) dx = \frac{1}{2} \tilde{C}_{\mu} \langle \tilde{l}^2 \rangle$$

and

$$\tilde{C}_{\mu} = \frac{2G}{\langle \tilde{l}^2 \rangle} \quad (\text{B12})$$

Finally

$$\iint \rho(\mathbf{r})\rho(\mathbf{r}') \frac{\varphi(\mathcal{O}_{\mathbf{r}}^{\mathbf{r}'})}{4\pi|\mathbf{r}' - \mathbf{r}|^2} d\mathbf{r} d\mathbf{r}' = \tilde{C}_{\mu} \int_0^{\infty} \tilde{\mu}(x) \left(\int_0^x \int_0^p \varphi(r)dr dp \right) dx, \quad \tilde{C}_{\mu} = \frac{2G}{\langle \tilde{l}^2 \rangle}. \quad (\text{B13})$$

APPENDIX C AVERAGE PATH LENGTH

Here is presented simple geometrical proof of equality of average path length inside a body and average chord length in uniform isotropic case.

On the Fig. 8a is depicted some path ABC with one kink inside a body. Let us also consider path FBD produced from ABC by central symmetry with respect to point B , Fig. 8b. The sum of lengths of two paths FBD and ABC is $|AB| + |BC| + |FB| + |BD|$ and coincides with sum for two chords AF and CD Fig. 8c.

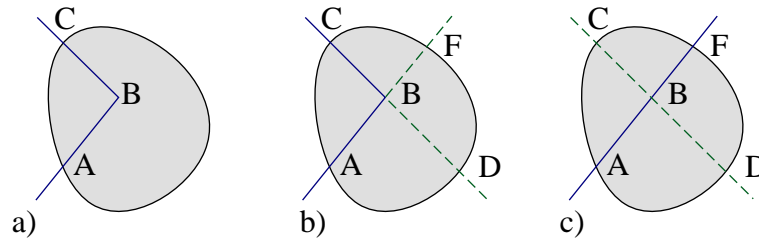


Figure 8: Illustration for average path length

This method let us get rid of one kink and after few such steps to consider average length of straight lines instead of paths. The necessary condition for such a proof — is isotropic and uniform distribution of paths to ensure equal probability (density) of ABC and FBD .

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